Integral Lipat Tiga

- Untuk fungsi tiga variabel.
- Analog dengan integral lipat dua, integral lipat tiga pada daerah

$$B = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

- Jika fungsi f kontinu pada daerah B, maka

$$\iiint_B f(x, y, z)\,dv = \int_a^b \int_c^d \int_r^s f(x, y, z)\,dz\,dy\,dx$$

- Teorema Fubini juga berlaku.

Hitung: $\iiint\limits_B 8xyz\,dV\,, \quad B=[2,3]\times[1,2]\times[0,1]$

Jawab: $\iiint\limits_B 8xyz\,dV = \int_1^2\int_2^3\int_0^1 8xyz\,dz\,dx\,dy$

$$= \int_1^2\int_2^3 4xyz^2\Big|_0^1\,dx\,dy$$

$$= \int_1^2\int_2^3 4xy\,dx\,dy$$

$$= \int_1^2 2x^2y\Big|_2^3\,dy$$

$$= \int_1^2 10y\,dy$$

$$= 5y^2\Big|_1^2$$

$$= 15$$

✓ Seperti halnya integral lipat dua, integral lipat tiga dapat juga berlaku pada daerah umum *E*.

1. Jenis I $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$

Jadi

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz \right] dA.$$

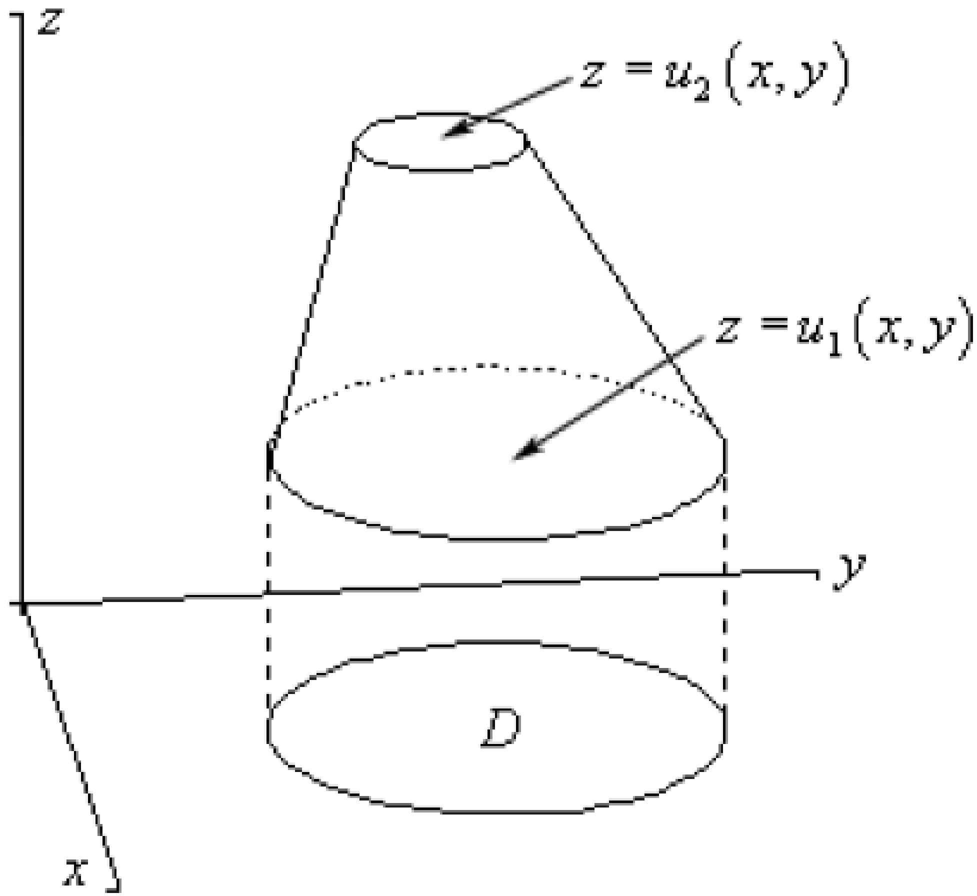
Daerah *D* dapat berupa (Ingat integral lipat dua):

1). Segiempat, $D = \{(x,y) \mid a \le x \le b, c \le y \le d\}$.

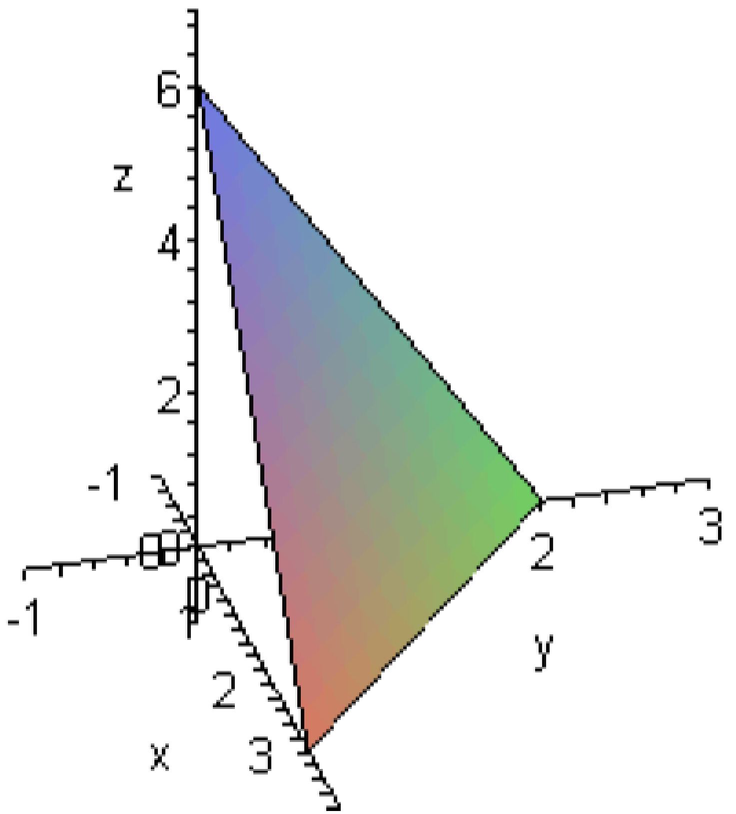
2). Daerah jenis I, $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$.

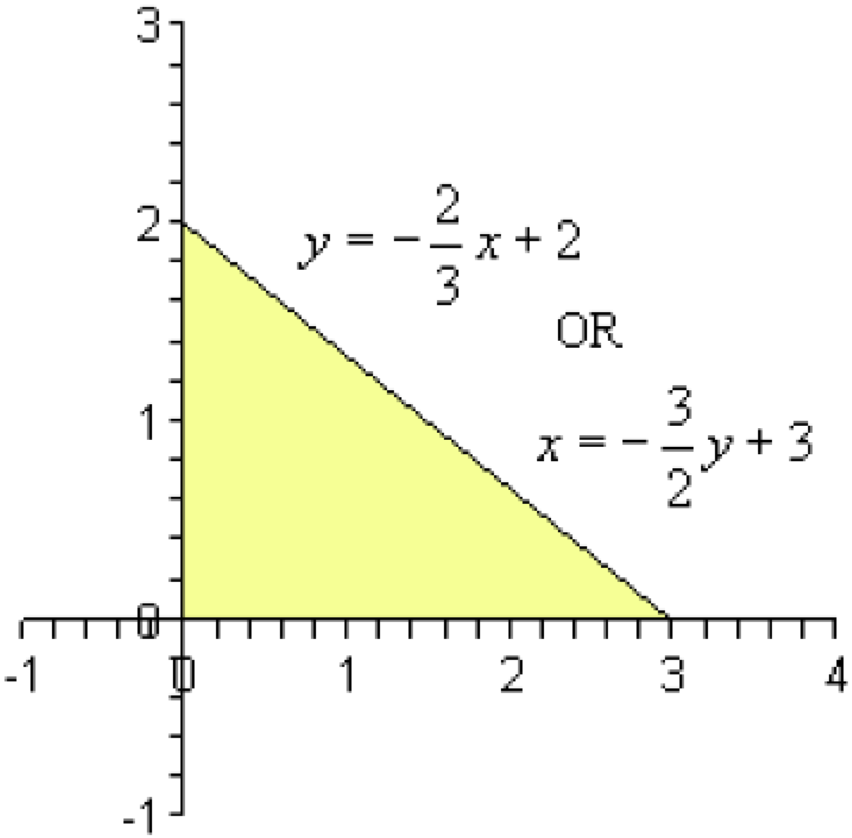
3). Daerah jenis II, $D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$.

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$



Contoh: Hitung $\iiint_E 2x\,dV$ dimana E adalah daerah di bawah bidang $2x + 3y + z = 6$ pada kuadran pertama





$$0 \le z \le 6 - 2x - 3y$$

$$0 \le x \le 3$$

$$0 \le y \le -\frac{2}{3}x + 2$$

$$0 \le x \le -\frac{3}{2}y + 3$$

$$0 \le y \le 2$$

$$\iiint_E 2x\,dV = \iint_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA$$

$$= \iint_D 2xz\Big|_0^{6-2x-3y}\,dA$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} 2x\left(6-2x-3y\right) dy\,dx$$

$$= \int_0^3 \left(12xy - 4x^2y - 3xy^2\right)\Big|_0^{-\frac{2}{3}x+2} dx$$

$$= \int_0^3 \frac{4}{3}x^3 - 8x^2 + 12x\, dx$$

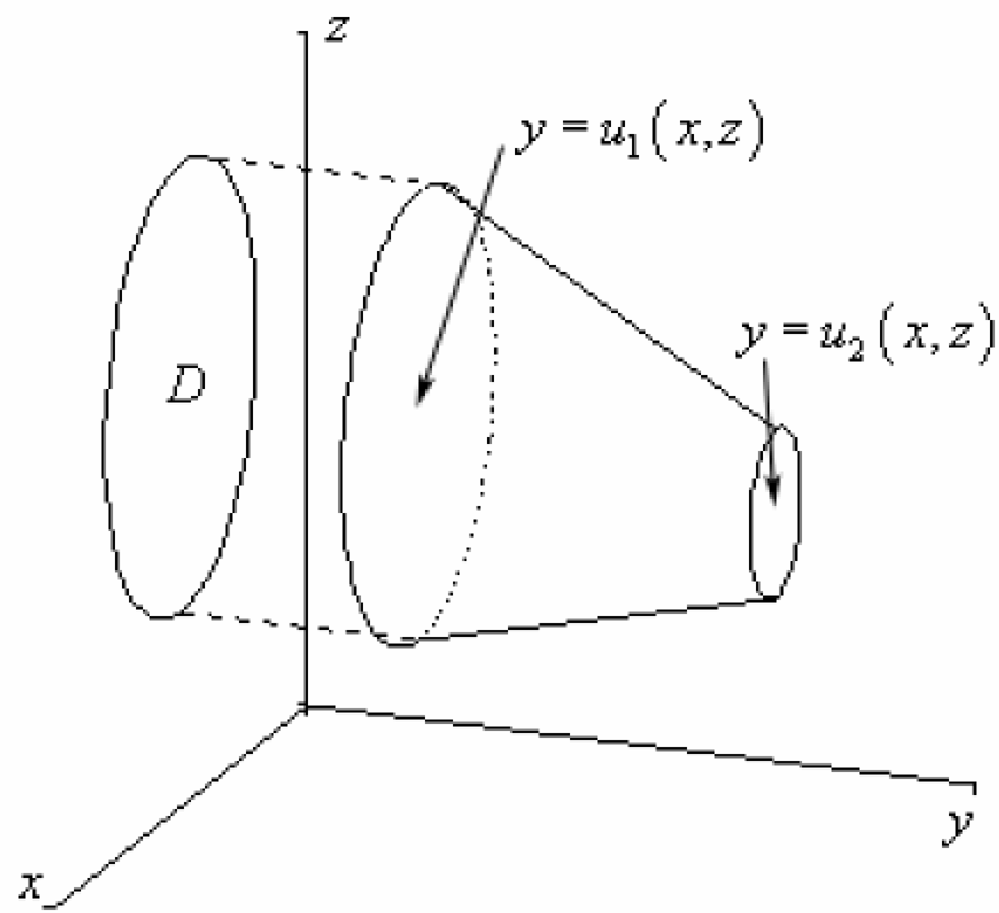
$$= \left(\frac{1}{3}x^4 - \frac{8}{3}x^3 + 6x^2\right)\Big|_0^3$$

$$= 9$$

2. Jenis II

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

D dapat berupa (1), (2) dan (3) seperti di jenis I.

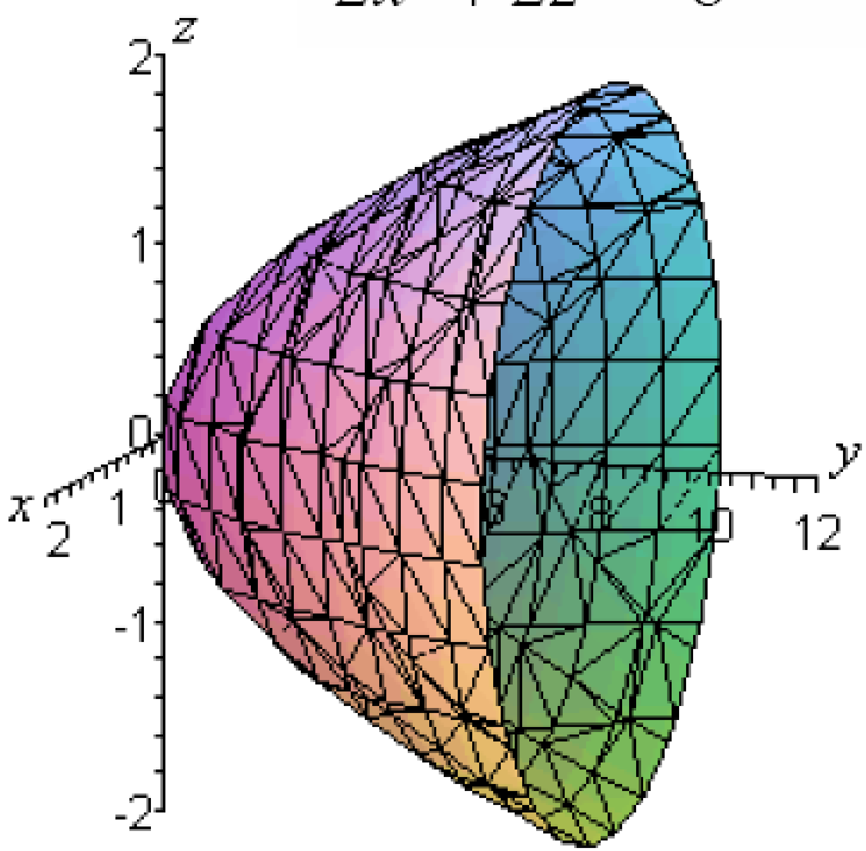


Contoh: **Hitung** $\iiint_E \sqrt{3x^2 + 3z^2}\, dV$

dimana E adalah benda pejal yang dibatasi oleh

$y = 2x^2 + 2z^2$ **dan bidang y = 8**

$$2x^2 + 2z^2 = 8 \quad \Rightarrow \quad x^2 + z^2 = 4$$



$$x = r\cos\theta$$
$$z = r\sin\theta$$

$$2x^2 + 2z^2 \le y \le 8$$
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\,dy\right] dA$$

$$= \iint_D \left(y\sqrt{3x^2+3z^2}\right)\Big|_{2x^2+2z^2}^{8} dA$$

$$= \iint_D \sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) dA$$

$$\sqrt{3\left(x^2+z^2\right)}\left(8-\left(2x^2+2z^2\right)\right) = \sqrt{3r^2}\left(8-2r^2\right)$$

$$= \sqrt{3}\ r\left(8-2r^2\right)$$

$$= \sqrt{3}\left(8r-2r^3\right)$$

$$\iiint_E \sqrt{3x^2+3z^2}\,dV = \iint_D \sqrt{3}\,\left(8r-2r^3\right)dA$$

$$= \sqrt{3}\int_0^{2\pi}\int_0^2 \left(8r-2r^3\right) r\,dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi} \left(\frac{8}{3}r^3 - \frac{2}{5}r^5\right)\Bigg|_0^2 d\theta$$

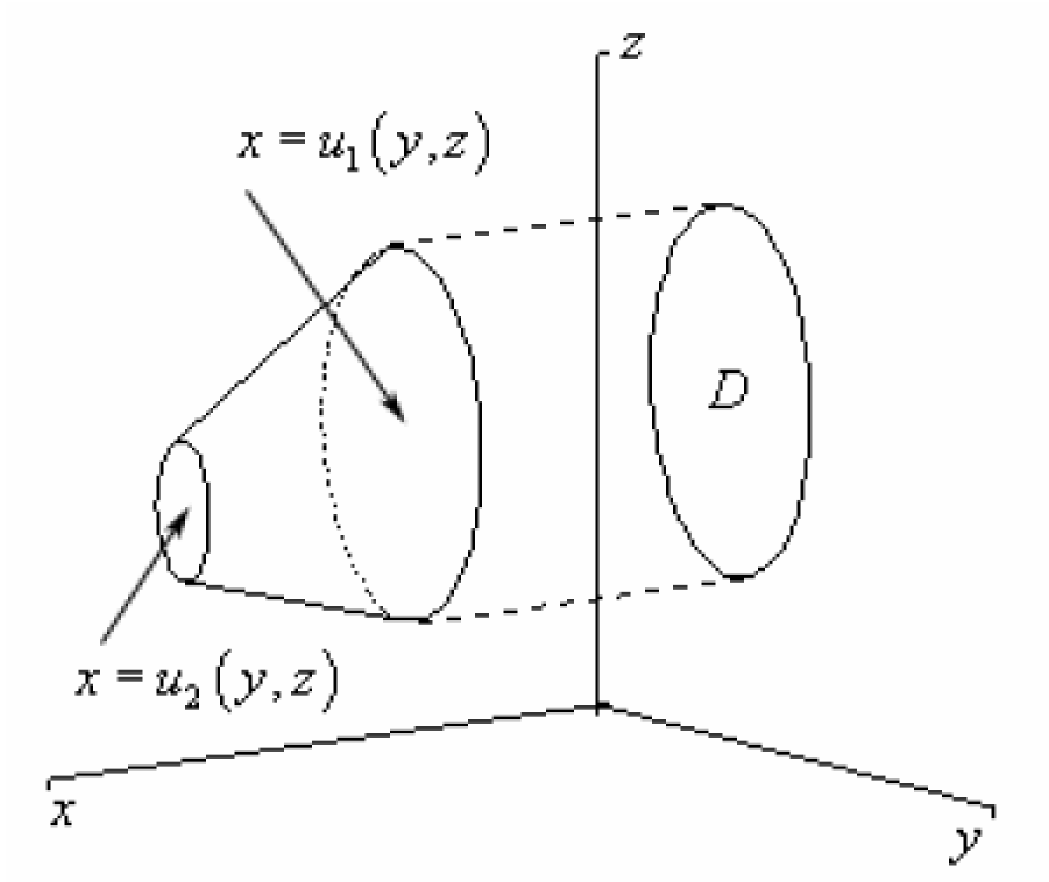
$$= \sqrt{3}\int_0^{2\pi} \frac{128}{15}\,d\theta$$

$$= \frac{256\sqrt{3}\,\pi}{15}$$

3. Jenis III

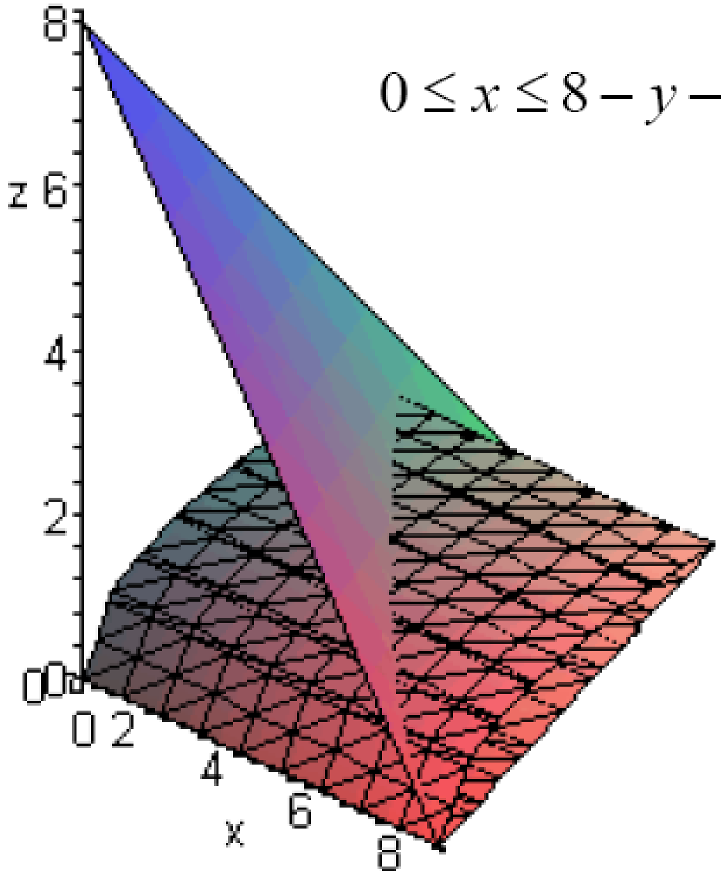
$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \le x \le u_2(y, z)\}$$

D dapat berupa (1), (2) dan (3) seperti di Jenis I.

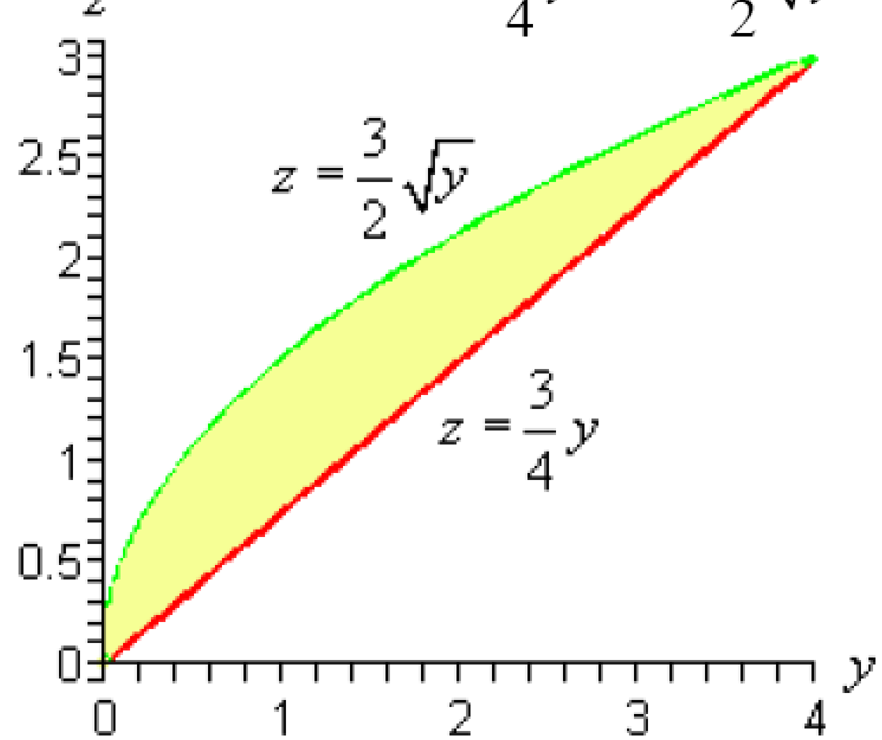


Contoh: **Hitung volume daerah yang terletak di belakang bidang $x+y+z=8$ dan di depan bidang yz yang dibatasi oleh $z=\frac{3}{2}\sqrt{y}$ dan $z=\frac{3}{4}y$**

$$0 \le x \le 8-y-z$$



$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$



$$0 \le y \le 4$$

$$V = \iiint\limits_E dV = \iint\limits_D \left[\int_0^{8-y-z} dx \right] dA$$

$$= \int_0^4 \int_{3y/4}^{3\sqrt{y}/2} 8 - y - z \, dz \, dy$$

$$= \int_0^4 \left(8z - yz - \frac{1}{2} z^2 \right) \Bigg|_{\frac{3y}{4}}^{\frac{3\sqrt{y}}{2}} dy$$

$$= \int_0^4 12y^{\frac{1}{2}} - \frac{57}{8} y - \frac{3}{2} y^{\frac{3}{2}} + \frac{33}{32} y^2 \, dy$$

$$= \left(8y^{\frac{3}{2}} - \frac{57}{16} y^2 - \frac{3}{5} y^{\frac{5}{2}} + \frac{11}{32} y^3 \right) \Bigg|_0^4 = \frac{49}{5}$$

16.8 Integral Lipat Tiga dalam Koordinat Silinder dan Koordinat Bola.

<u>Koordinat Silinder</u>

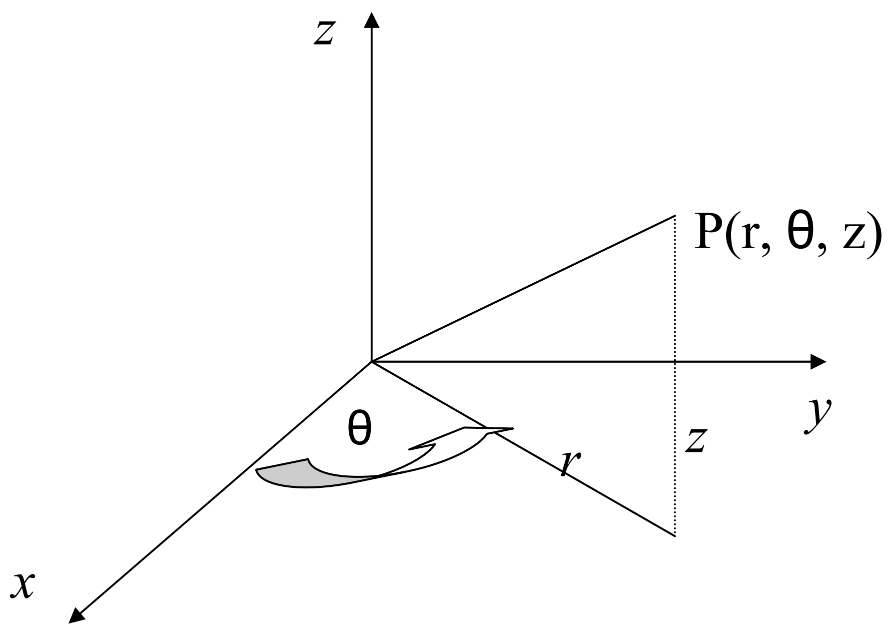
Dalam koordinat silinder, titik P(*x*,*y*,*z*) dikonversi ke titik P(*r*,θ,*z*).

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$$dV = r\,dz\,dr\,d\theta$$



Misal f kontinu pada E dan

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

dengan D diberikan dalam koordinat polar oleh

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

$$\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{h_1(\theta}^{h_2(\theta} f(x, y, z)\,dz \right] dA$$

dikonversi dari koordinat siku - siku ke koordinat silinder sebagai berikut :

$$\iiint_E f(x,y,z)\,dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Contoh: **Hitung** $\iiint_E y\,dV$ **dimana E adalah daerah di bawah bidang** $z = x + 2$ **dan di atas bidang *xy* serta di antara silinder** $x^2 + y^2 = 1$ **dan** $x^2 + y^2 = 4$

Jawab: $0 \le z \le x + 2 \quad \Rightarrow \quad 0 \le z \le r\cos\theta + 2$

$0 \le \theta \le 2\pi \qquad\qquad 1 \le r \le 2$

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} (r\sin\theta)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta(r\cos\theta + 2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^2 \sin\theta \left(r\cos\theta + 2 \right) dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^2 \frac{1}{2} r^3 \sin\left(2\theta\right) + 2r^2 \sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{8} r^4 \sin\left(2\theta\right) + \frac{2}{3} r^3 \sin\theta \right)\Bigg|_1^2 d\theta$$

$$= \int_0^{2\pi} \frac{15}{8} \sin\left(2\theta\right) + \frac{14}{3} \sin\theta\, d\theta$$

$$= \left(-\frac{15}{16} \cos\left(2\theta\right) - \frac{14}{3} \cos\theta \right)\Bigg|_0^{2\pi}$$

$$= 0$$

Contoh: Konversi integral berikut ke koordinat silindris

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$$

Jawab:

$$-1 \le y \le 1 \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$0 \le x \le \sqrt{1-y^2} \qquad 0 \le r \le 1$$

$$x^2+y^2 \le z \le \sqrt{x^2+y^2} \qquad r^2 \le z \le r$$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$$

$$=\int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} r\left(r\cos\theta\right)\left(r\sin\theta\right)z\,dz\,dr\,d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} zr^3\cos\theta\sin\theta\,dz\,dr\,d\theta$$

Koordinat Bola

Perhatikan Δ OPP' :

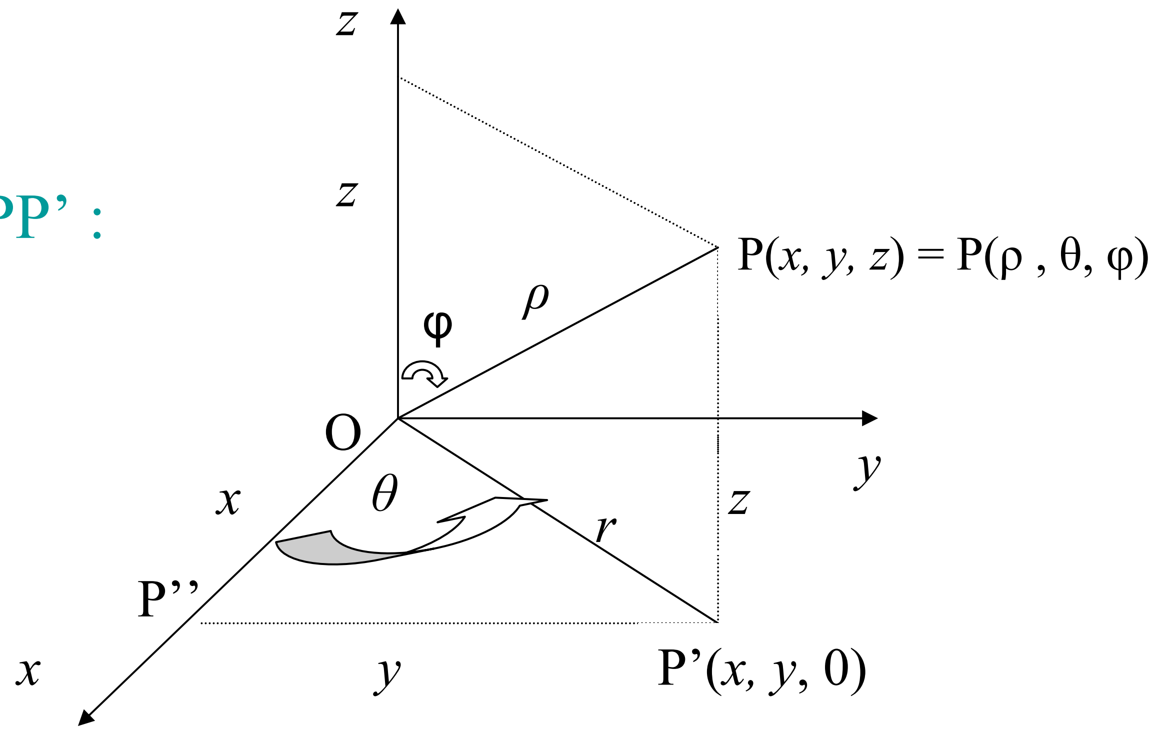
$z = \rho \cos \varphi$

$r = \rho \sin \varphi$

$x = \rho \sin \varphi \cos \theta$

$y = \rho \sin \varphi \sin \theta$

$x^2 + y^2 + z^2 = \rho^2$



$\rho \geq 0$ $\qquad 0 \leq \varphi \leq \pi$

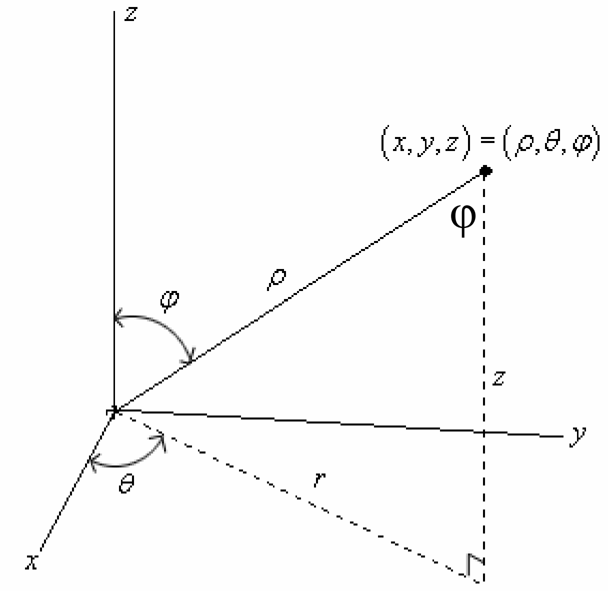
$$\text{Perhatikan } \Delta\, \mathrm{OP'P''}:$$

$$x = r\cos\theta \qquad ;\text{karena } r = \rho\sin\varphi, \text{ maka}$$

$$= \rho\sin\varphi\cos\theta$$

$$y = r\sin\theta \qquad ;\text{karena } r = \rho\sin\varphi, \text{ maka}$$

$$= r\sin\varphi\cos\theta$$



$$\rho^2 = r^2 + z^2 \qquad ;\Delta\, \mathrm{OPP'}$$

$$\underbrace{\quad = x^2 + y^2 + z^2}_{\text{Persamaan Bola dengan } \rho \text{ jari-jari bola}} \qquad ; r^2 = x^2 + y^2 \quad (\Delta\, \mathrm{OP'P''})$$

ρ : jarak titik asal ke P.

θ : sudut antara sumbu - x dengan OP'.

φ : sudut antara sumbu - z positif dengan OP.

$\rho \geq 0, 0 \leq \varphi \leq \pi.$

$$E = \{(\rho, \theta, \varphi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \delta \leq \varphi \leq \gamma\}$$

Dalam Koordinat Polar :

$$dA = r\, d\theta\, dr$$

Dalam Koordinat Bola (Keterangan pada halaman 489)

$$\Delta V_{ijk} = \underbrace{\Delta\rho}_{\text{tinggi}} \underbrace{(\rho_i \Delta\varphi)(\rho_i \sin\varphi_k \Delta\theta)}_{\text{Luas Alas}}$$

$$dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

Jadi, konversi dari koordinat siku-siku ke koordinat bola

$$\iiint_E f(x,y,z)\,dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

Contoh: **Hitung** $\iiint_E 16z\, dV$

dimana E adalah setengah bola $x^2 + y^2 + z^2 = 1$

bagian atas

Jawab:

$$0 \le \rho \le 1$$

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \frac{\pi}{2}$$

$$\iiint_E 16z\, dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\varphi \left(16\rho\cos\varphi\right) d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 8\rho^3 \sin\left(2\varphi\right) d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 2\sin\left(2\varphi\right) d\theta\, d\varphi$$

$$= \int_0^{\frac{\pi}{2}} 4\pi \sin\left(2\varphi\right) d\varphi$$

$$= -2\pi\cos\left(2\varphi\right)\Big|_0^{\frac{\pi}{2}}$$

$$= 4\pi$$

Contoh: konversi integral berikut ke koordinat bola

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz \, dx \, dy$$

Jawab:

$$0 \le y \le 3$$

$$0 \le x \le \sqrt{9 - y^2}$$

$$\sqrt{x^2 + y^2} \le z \le \sqrt{18 - x^2 - y^2}$$

$$0 \le \theta \le \frac{\pi}{2}$$

$$x^2 + y^2 + z^2 = 18$$

$$0 \le \rho \le \sqrt{18} = 3\sqrt{2}$$

$$\left(\sqrt{x^2+y^2}\right)^2+z^2=18$$

$$z^2+z^2=18$$

$$z^2=9$$

$$z=3$$

$$\rho\cos\varphi=3$$

$$3\sqrt{2}\cos\varphi=3$$

$$\cos\varphi=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \quad \Rightarrow \quad \varphi=\frac{\pi}{4} \quad \Rightarrow \quad 0\le\varphi\le\frac{\pi}{4}$$

$$z = r$$

$$\rho \cos\varphi = \rho \sin\varphi$$

$$1 = \tan\varphi \qquad \Rightarrow \qquad \rho = \frac{\pi}{4}$$

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz\, dx\, dy$$

$$= \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3\sqrt{2}} \rho^4 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

Tugas 1. Hitung $\iiint\limits_{E} 12xy^2z^3\,dV$

dengan $E = \{(x, y, z) \mid -1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2\}$

Tugas 2. Misalkan E adalah daerah pada oktan pertama yang dibatasi $y^2 + z^2 = 1, y = x$ dan $x = 0$.

Hitunglah $\iiint\limits_{E} z\,dV$.

Tugas 3. Gunakan integral lipat tiga untuk mencari volume benda pejal yang dibatasi silinder $x^2 + y^2 = 9$ dan bidang $z = 1$ dan $x + z = 5$.

Tugas 4.

Misalkan E terletak dalam silinder $x^2 + y^2 = 16$ dan di antara bidang – bidang $z =$ -5 dan $z = 4$.

Hitung $\iiint_E \sqrt{x^2 + y^2}\, d\mathrm{V}$.

Tugas 5.

Hitung $\iiint_E (x^3 + xy^2)\, d\mathrm{V}$, dengan E adalah benda pejal di oktan pertama yang terletak di bawah paraboloid $z = 1 - x^2 - y^2$.

Tugas 6.

Hitung $\iiint_E y\, d\mathrm{V}$, dengan E adalah benda pejal yang terletak diantara silinder - silinder $x^2 + y^2 = 1$ dan $x^2 + y^2 = 4$, di atas bidang - xy dan di bawah bidang $z = x + 2$.