Mathematical Analysis of Recursive Algorithm using Characteristic Equation

Homogeneous Linear Recurrences

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0,$$

- k and a_i are constant
- It's called "linear" because every term t_i appears only to the first power
- It's called "homogeneous" because the linear combination is equal to 0.

Homogeneous Linear Recurrences

- The following are homogeneous linear recurrence equations

$$7t_n - 3t_{n-1} = 0$$

$$6t_n - 5t_{n-1} + 8t_{n-2} = 0$$

$$8t_n - 4t_{n-3} = 0$$

Homogeneous Linear Recurrences

- The Fibonacci Sequence is defined as follows:

$$\boxed{\begin{aligned} t_n &= t_{n-1} + t_{n-2} \\ t_0 &= 0 \\ t_1 &= 1 \end{aligned}}$$

- Subtract t_{n-1} and t_{n-2} from both sides, we get homogeneous linear recurrence equation

$$t_n - t_{n-1} - t_{n-2} = 0,$$

Characteristic Equation

- Definition

The ***characteristic equation*** for the homogeneous linear recurrence equation with constant coefficients

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0$$

is defined as

$$a_0 r^k + a_1 r^{k-1} + \cdots + a_k r^0 = 0.$$

Characteristic (cont.)

- Example

The characteristic equation for the recurrence appears below it:

$$5t_n - 7t_{n-1} + 6t_{n-2} = 0$$

$$5r^2 - 7r + 6 = 0.$$

We use an arrow to show that the order of the characteristic equation is k (in this case, 2).

Characteristic (cont.)

- Theorem. Let the homogeneous linear recurrence equation with constant coefficients

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0$$

be given. If its characteristic equation

$$a_0 r^k + a_1 r^{k-1} + \cdots + a_k r^0 = 0$$

has k distinct solutions $r_1, r_2, \ldots, r_k$, then the only solutions to the recurrence are

$$t_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n,$$

Characteristic (cont.)

- Consider the following recurrence

$$
\begin{cases}
t_n - 3t_{n-1} - 4t_{n-2} = 0 & \text{for } n > 1 \\
t_0 = 0 \\
t_1 = 1
\end{cases}
$$

1. Obtain the characteristic equation

$$
\begin{aligned}
t_n - 3t_{n-1} - 4t_{n-2} &= 0 \\
r^2 - 3r - 4 &= 0.
\end{aligned}
$$

Characteristic (cont.)

2. Solve the characteristic equation

$$r^2 - 3r - 4 = (r - 4)(r + 1) = 0.$$

3. Apply the theorem to get the general solution:

$$t_n = c_1 4^n + c_2(-1)^n.$$

4. Determine the values of the constants

$$t_0 = 0 = c_1 4^0 + c_2(-1)^0$$
$$t_1 = 1 = c_1 4^1 + c_2(-1)^1.$$

Characteristic (cont.)

The solution to this system is c_1 =1/5, c_2 =-1/5

5. Substitute the constant into the general solution to obtain the particular solution:

$$t_n = \frac{1}{5}4^n - \frac{1}{5}(-1)^n.$$

Non-Homogeneous Linear Recurrences

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = f(n)$$

- k and a_i are constant, $f(n)$ is a function other than the zero function.
- Develop a method for solving the common case

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b^n p(n),$$

where b is a constant and $p(n)$ is a polynomial in n.

Characteristic (cont.)

- Theorem. A nonhomogeneous linear recurrence of the form can be transformed into homogeneous linear recurrence

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b^n p(n),$$

where *d* is the degree of *p(n)*.

$$(a_0 r^k + a_1 r^{k-1} + \cdots + a_k)(r - b)^{d+1} = 0,$$

- Theorem. Let r be a root of multiplicity m of the characteristic equation for a homogeneous linear recurrence with constant coefficients. Then

$$t_n = r^n, \quad t_n = nr^n, \quad t_n = n^2 r^n, \quad t_n = n^3 r^n, \quad \cdots, \quad t_n = n^{m-1} r^n$$

are all solutions to the recurrence.

Characteristic (cont.)

- Solve the following recurrence

$$
\boxed{\begin{aligned} & t_n - 3t_{n-1} = 4^n(2n+1) \qquad \text{for } n > 1 \\ & t_0 = 0 \\ & t_1 = 12 \end{aligned}}
$$

1. Obtain the characteristic eq for the corresponding homogeneous eq:

$$
\begin{aligned} t_n - 3t_{n-1} &= 0 \\ r^1 - 3 &= 0. \end{aligned}
$$

Characteristic (cont.)

2. Obtain a term from the nonhomogeneous part of the recurrence

   the term is

$$\underset{\substack{\uparrow\\ b}}{4}^{n}\left(2n^{\overset{\substack{d\\ \downarrow}}{1}} + 1\right).$$

$$(r - b)^{d+1} = (r - 4)^{1+1}.$$

Characteristic (cont.)

3. Apply theorem to obtain the characteristic eq.

$$(r-3)(r-4)^2.$$

4. Solve the characteristic equation

$$(r-3)(r-4)^2=0.$$

the roots are r=3 and r=4, and the root r=4 has multiplicity 2

Characteristic (cont.)

5. Apply theorem to get the general solution

$$t_n = c_1 3^n + c_2 4^n + c_3 n 4^n.$$

We must find another initial condition. Because

and t_1 =12,

$$t_2 - 3t_1 = 4^2(2 \times 2 + 1).$$

$$t_2 = 3 \times 12 + 80 = 116.$$

Characteristic (cont.)

6. Determine the value of the constants
7. Substitute the constants into the general solution to obtain

$$t_n = 20(3^n) - 20(4^n) + 8n4^n.$$

Exercises

t_n-5t_{n-1} + 6t_{n-2} = 0, for n > 1

$t_0 = 0$

t_1=1

$t_n - 3t_{n-1} = 4^n$, for n > 1

$t_0 = 0$

T_1=4

Reference

- Neapolitan Naimipour, Foundation of Algorithm, DC Health & Company, 1996